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LETTER TO THE EDITOR

On scaling relations in growth models for percolating clusters and diffusion fronts

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Abstract. Employing analogies between the growth of the incipient percolating cluster and the growth of a diffusion front, we develop a scaling theory for this latter model. Our scaling assumptions support the Sapoval–Rosso–Gouyet conjecture, $d_H = 1 + 1/\nu$, which relates, in two dimensions, the fractal dimension d_H of the hull of the incipient percolating cluster to the correlation length exponent ν in percolation, and yield relations between the static and dynamic exponents of the growth of a diffusion front.

It has been recently conjectured by Sapoval *et al* (SRG) (1985) that in two dimensions the fractal dimension d_H of the hull of the infinite percolating cluster is simply related to the correlation length ν via

$$d_H = 1 + 1/\nu. \tag{1}$$

The conjecture was made on the basis of numerical data obtained for the time evolution of a diffusion front on a square lattice. In the following we will give scaling arguments which strongly support this conjecture and, moreover, relate the typical behaviour of the diffusion front to the typical behaviour in percolation growth models.

First let us recall the characteristic features of the diffusion front studied by SRG. In general, a diffusion front can be generated as follows. At time 0 the first row ($x \equiv 0$) of the (x, y) lattice is occupied by particles. Then each particle can jump to nearest-neighbour sites, provided the site is not occupied by another particle and is inside the lattice. The time is increased by one after all particles in the system have tried to make a jump and then the first row in the system is refilled again. To avoid this computer time consuming procedure SRG used the following trick. The density distribution $p(x, t)$ of the diffusing particles along the x direction at time t is determined by the error function:

$$p(x,t) = P_0(x/l_d(t)) = \operatorname{erfc}\left(\frac{x}{l_d(t)}\right) = 1 - \frac{2}{\pi^{1/2}}\int_0^{x/l_d(t)} d\eta \exp(-n^2)$$

where $l_d(t) \equiv 2(Dt)^{1/2}$ is the diffusion length. Then the diffusion process was simulated by SRG by distributing particles randomly on each row of length L with probability $P_0(x/l_d(t))$. This procedure indeed simulates the real diffusion process as long as

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many-particle correlations can be neglected, which is a reasonable assumption for a hard-core system.

The diffusion front is defined as the frontier between the 'land' of occupied sites and the 'seat' of empty sites, which is connected with the occupied sites at $x=0$ and the empty sites at $x=\infty$ through neighboured (nearest neighbours and next-nearest neighbours) occupied and empty sites, respectively. It was then shown that this diffusion front has asymptotically ($l_d \to \infty$) the following properties:

(i) It has the fractal dimension of the hull of the incipient percolating cluster, which had been estimated numerically by Voss (1984)

$$d_H \simeq 1.76 \pm 0.02.$$

(ii) Its mean width (along the x direction) follows a power law as a function of l_d:

$$\sigma_f \sim l_d^{\alpha_\sigma} \qquad \alpha_\sigma = \nu/(\nu+1) \tag{2}$$

where ν is the exponent of the correlation length in percolation, $\nu = \frac{4}{3}$.

(iii) The number of particles N on the diffusion front satisfies

$$N \sim l_d^{\alpha_N} \qquad \alpha_N = (d_H - 1)\nu/(\nu+1). \tag{3}$$

From the numerical observation that $\alpha_\sigma + \alpha_N \simeq 1$ the conjecture (1) was made.

Let us now consider analogies between the growth of this diffusion front and percolation growth models on the basis of which we will perform a scaling analysis for the evolution of the diffusion front.

The kinetics of growing percolation clusters is described by the evolution of growth sites which denote those sites of the lattice where the cluster can grow (Leyvraz and Stanley 1983, Ben Avraham and Havlin 1982). Roughly speaking, the G sites define the unblocked surface of the growing cluster. The number G of G sites surrounding a cluster of s sites follows a power law

$$G \sim s^{d_G/d_F} \tag{4}$$

where d_F is the fractal dimension of the cluster and d_G determines the fractal dimension of the volatile G-site fractal. In contrast to d_F d_G depends on the way the incipient percolating cluster is generated (Bunde *et al* 1984). The width of the distribution of the G sites, Δ, is assumed to scale as the cluster radius (Stanley *et al* 1984, Bunde *et al* 1984) and therefore is related to s via

$$s \sim \Delta^{d_F} \tag{5a}$$

and thus

$$G \sim \Delta^{d_G}. \tag{5b}$$

Regarding the growth of a diffusion front, the number of particles N in the diffusion front along the x axis is analogous to the number of growth sites G, σ_f corresponds to Δ, and s is equivalent to l_d. Therefore we postulate analogous scaling relations to the growth of the diffusion front:

$$l_d \sim \sigma_f^{d_H} \tag{6a}$$

and

$$N \sim l_d^{d_N/d_H} \sim \sigma_f^{d_N} \tag{6b}$$

which defines d_N in analogy to d_G.

Comparison of ($6a$) and ($6b$) with (2) and (3) now yields:

$$\frac{d_N}{d_{\mathrm{H}}} = \alpha_N = \frac{\nu}{\nu+1}(d_{\mathrm{H}} - 1) \tag{7a}$$

$$\frac{1}{d_{\mathrm{H}}} = \alpha_\sigma = \frac{\nu}{\nu+1} \tag{7b}$$

and

$$d_N = d_{\mathrm{H}} - 1. \tag{7c}$$

The last relation is obvious and follows directly from the translational invariance along the y axis. The width L of the samples is assumed to be large enough to make the finite-size effect negligible. Equation ($7c$) is analogous to the recent Aharony–Stauffer arguments (Aharony and Stauffer 1984) for percolating clusters grown by a random walk. While the conjecture could not be verified for percolation (Stanley *et al* 1984) it holds here. Equation ($7b$) relates the fractal dimension of the hull of the percolating cluster to the exponent ν of correlation length. Taking the accepted value $\nu = \frac{4}{3}$ we find

$$d_{\mathrm{H}} = \tfrac{7}{4}$$

which is in agreement with the numerical data. Comparing this value with the value $d_{\mathrm{F}} = \frac{91}{48}$, for the fractal dimension of the whole percolating cluster, we find simply

$$d_{\mathrm{H}} = \tfrac{12}{13} d_{\mathrm{F}}.$$

In other words, the hull or external perimeter of a large two-dimensional percolation cluster at the threshold increases as (number of sites)$^{12/13}$. This exponent $\frac{12}{13} = 0.923$ is in good agreement with numerical data (Leath and Reich 1978, Kremer and Lyklema 1984, Weinrib and Trugman 1984). Our theory is based on analogies between percolation clusters and diffusion fronts.

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